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Quants

The Limits of Linear Ranking:

What Geometry Allows—and Forbids

This is the fourth note in a series on quantamental stock rankings. The [first note (Ranking Before Prediction)](), the second one was [(Why Learning Factor Weights Is an Ill-Posed Inverse Problem)]() and the third ([Ranking as Selection: From N- grams and Tokens to Equity Universes]()) . Sorry if this note is a bit heavy on math, but it discusses non obvious things that exclude purely linear methods from a set of instruments able to achieve valid results in stock selection based on rankings.

A small example that refuses to go away

Consider three stocks, each described by two standardized factors:

- Z_PE (cheapness; lower is better., *or negative Z_PE then higher is better*)
- Z_EPSG (earnings growth; higher is better)

Let their factor vectors be:

- Stock A: a = (1, 1)
- Stock B: b = (−1, 1)
- Stock C: c = (0, 1)

Assume a linear scoring rule: s(z) = x1 * Z_PE + x2 * Z_EPSG

The scores are:

- s(A) = x1 + x2
- s(B) = −x1 + x2
- s(C) = x2

Notice the identity: s(C) = (s(A) + s(B)) / 2

This holds for all real x1, x2.

As a result, some orderings are impossible. For example, there is no choice of (x1, x2) such that: s(A) > s(B) > s(C)

This is not a numerical accident. It is a geometric constraint.

This small example captures, in its simplest form, a phenomenon that reappears at scale in real factor models.

What actually failed here?

Nothing failed computationally.

We did not lack data.
We did not choose the wrong optimizer.
We did not search weights poorly.

The desired ordering was **geometrically infeasible** under a linear scoring rule.

Why?

Because Stock C lies exactly between A and B in factor space:

$c = 0.5 * a + 0.5 * b$

Linear functions preserve convex combinations. A midpoint cannot be ranked strictly above or strictly below both endpoints by any linear functional.

This observation generalizes.

The general convexity constraint

Let $z1, z2, \ldots, zk$ be points in R^n.

If a point $z0$ can be written as a convex combination:

$z0 = \lambda 1 \mathit{z1} + \mathit{\lambda 2} z2 + \ldots + \lambda k * zk$
with all $\lambda i \geq 0$ and $\mathrm{sum}(\lambda i) = 1$,

then for any linear scoring rule:

$s(z) = w \cdot z$

we have:

$\min s(zi) \leq s(z0) \leq \max s(zi)$

Therefore:

- z0 cannot be the unique top-ranked point
- z0 cannot be the unique bottom-ranked point

This is a **hard constraint**, independent of sample size or noise.

In factor language:

> If a stock's factor vector lies in the convex hull of others, no linear factor weighting can make it an extreme.

From three stocks to many: where permutations come from

Now consider m stocks with factor vectors $z_1, z_2, \ldots, z_m$ in $\mathbb{R}^n$.

A linear scoring rule assigns scores:

$s_i = w \cdot z_i$

A strict ranking corresponds to a strict ordering of these scores.

Two stocks i and j tie when:

$w \cdot (z_i - z_j) = 0$

This equation defines a hyperplane in weight space.

For m stocks, there are $m(m-1)/2$ such hyperplanes. These hyperplanes partition weight space into regions (cones). Inside each region, the induced ranking is constant.

Therefore:

> **Each realizable ranking corresponds to a region of a hyperplane arrangement in weight space.**

This observation is standard in geometric combinatorics and goes back at least to Thomas Cover's work on linearly inducible orderings.

Why you cannot get all permutations

The crucial consequence is this:

The number of regions formed by N hyperplanes in $\mathbb{R}^n$ is at most:

$\sum_{k=0}^{n} C(N, k)$

Here, $N = m(m-1)/2$.

For fixed n, this quantity grows polynomially in m.

But the total number of possible rankings of m items is:

$m!$ which grows much faster than any polynomial.

Therefore:

> For fixed factor dimension n, a linear scoring rule can realize only a tiny fraction of all possible rankings once m is large.

This is not a defect of factor models. It is a mathematical ceiling.

Why S3 feels special—and why it misleads

(S3 is a group of all permutations on 3 objects)

With three points in two dimensions:

- If the points are not collinear, all six permutations are realizable.
- If the points are collinear, only two are (forward and reverse).

Note, that in the example in the entry paragraph all 3 points are collinear. e.g. $y=1$ for all 3.

This makes S3 feel deceptively flexible.

The illusion disappears immediately for S4 and beyond. Even in two dimensions, four points already impose ordering constraints. In higher dimensions the situation improves, but only slowly. The factorial growth of permutations always wins.

This explains why:

- Some rankings never appear in backtests
- Monte Carlo searches fail silently
- Weight tuning hits invisible walls

What Monte Carlo is actually doing

When we randomly sample weight vectors w, we are not “searching for optimal weights.”

We are **probing the geometry** of the feasible ranking space.

Each random w lands in one of the cones defined by the hyperplane arrangement. All w inside the same cone produce the same ordering.

Scaling w by any positive constant leaves the ordering unchanged. Only direction matters.

Therefore:

- Many distinct weight vectors produce identical rankings
- The natural object is a **region**, not a point
- Absence of a ranking under Monte Carlo usually means infeasibility, not bad luck

This reframes Monte Carlo as an exploratory geometric tool, not an optimizer.

Why nonlinearity changes the picture

Linear scoring preserves convexity.

Nonlinear scoring does not.

If we modify the score to include a nonlinear term, for example:

$s(z) = w \cdot z + \alpha * (Z\_PE)^2$

then midpoints are no longer preserved. Points inside the convex hull can become extreme.

This explains why:

- interaction terms
- regime splits
- decision trees
- piecewise models

often feel disproportionately powerful in practice.

They do not “find better weights.”
They **reshape the geometry**.

What this means for real equity models

In realistic settings:

- Factor normalization increases convexity
- Correlated factors reduce effective dimension
- Industry-relative scoring compresses variation

As a result:

- Many stocks occupy a dense interior region
- Extreme rankings are fragile
- Stability matters more than exact order

Linear factor models are therefore best understood as **controlled projections**, not universal ranking engines.

Their limitations are geometric, not statistical.

Geometry before optimization

The central lesson of this note is simple but often missed:

> Linear ranking models do not fail because we estimate weights poorly.
> They fail because some rankings are not representable at all.

Once this is understood, several long-standing puzzles become clearer:

- Why factor weights are unstable
- Why multiple weight sets “work” equally well
- Why neutral stocks dominate universes
- Why adding modest nonlinearity has large effects

Geometry comes before optimization.

And ranking, not prediction, is the object that geometry constrains.

Appendix: Mathematical Background and References

This series touches several areas of mathematics and statistical theory that are usually studied separately. Below is a brief guide to the most relevant strands and how they connect to the ideas discussed in these notes.

A. Linearly inducible orderings and hyperplane arrangements

The foundational observation behind linear ranking systems is that every pairwise comparison between two objects defines a hyperplane in weight space. These hyperplanes partition the space of possible weight vectors into regions, and each region corresponds to a fixed ranking.

This viewpoint originates in work by **Thomas M. Cover**, who studied the number of distinct orderings that can be induced by linear functionals. His results show that, for fixed dimension, the number of realizable orderings grows polynomially with the number of objects—far more slowly than the factorial growth of all possible permutations.

- **Cover, T. M. (1967)**
  The Number of Linearly Inducible Orderings of Points in d-Space
  IEEE Transactions on Electronic Computers

This paper provides the conceptual backbone for understanding why many desired rankings are simply unreachable under linear scoring rules.

Closely related is the theory of **hyperplane arrangements**, which formalizes how collections of hyperplanes partition space and how many regions they create.

- **Zaslavsky, T. (1975)**
  Facing Up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes
  Memoirs of the American Mathematical Society

- **Stanley, R. P. (2004)**
  An Introduction to Hyperplane Arrangements
  Geometric Combinatorics Lecture Notes

These results explain why linear ranking models naturally produce equivalence classes of weights rather than unique solutions.

B. Convex geometry and infeasible rankings

A recurring theme in these notes is that linear scoring preserves convex combinations. Points lying inside the convex hull of others cannot be made extreme by any linear functional. This is a basic but powerful result from convex geometry and linear programming.

Standard references include:

- **Rockafellar, R. T. (1970)**
  Convex Analysis
  Princeton University Press

This convexity constraint underlies many of the “impossible ordering” examples discussed in Blog 4.

C. Oriented matroids and allowable sequences

When studying rankings induced by sweeping a linear functional across a point configuration, the natural combinatorial structure that emerges is an **oriented matroid**. Oriented matroids capture the sign patterns of linear functionals and formalize which orderings are combinatorially consistent.

In discrete geometry, the sequence of permutations observed as a direction rotates is known as an **allowable sequence**.

Key references:

- **Goodman, J. E., & Pollack, R. (1984)**
  Semispaces of Configurations, Cell Complexes of Arrangements
- **Björner, A., Las Vergnas, M., Sturmfels, B., White, N., & Ziegler, G. (1999)**
  Oriented Matroids
  Cambridge University Press

This literature provides a rigorous framework for understanding ranking constraints beyond low-dimensional examples.

D. Cover's work on portfolio construction and factor geometry

In addition to his work on orderings, **Cover** made foundational contributions to portfolio theory that resonate strongly with factor-based ranking systems.

In particular, Cover studied portfolio selection as a geometric and information-theoretic problem, emphasizing relative performance, invariance, and long-run growth rather than point forecasts.

- **Cover, T. M. (1991)**
  Universal Portfolios
  Mathematical Finance
- **Cover, T. M., & Thomas, J. A. (2006)**
  Elements of Information Theory
  Wiley

Cover's perspective reinforces a central theme of these notes:
selection, relative ordering, and robustness often matter more than precise estimation.

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